

HYPERSURFACES WITH NONNEGATIVE SCALAR CURVATURE

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ABSTRACT. We show that closed hypersurfaces in Euclidean space with nonnegative scalar curvature are weakly mean convex. In contrast, the statement is no longer true if the scalar curvature is replaced by the k -th mean curvature, for k greater than 2, as we construct the counterexamples for all k greater than 2. Our proof relies on a new geometric inequality which relates the scalar curvature and mean curvature of a hypersurface to the mean curvature of the level sets of a height function. By extending the argument, we show that complete non-compact hypersurfaces of finitely many regular ends with nonnegative scalar curvature are weakly mean convex, and prove a positive mass theorem for such hypersurfaces.

1. INTRODUCTION

For n -dimensional hypersurfaces in Euclidean space, it is natural to understand the relations between the intrinsic curvature and the extrinsic curvature. In 1897, Hadamard [7] proved that a closed (i.e., compact without boundary) surface embedded in \mathbb{R}^3 of positive Gaussian curvature is a convex surface. Hadamard's result was extended by Stoker [22] to the complete non-compact case.

In contrast to the strict inequality assumption on the curvature, the non-strict inequality case is more subtle. About sixty years later, Chern–Lashof [5] was able to prove that Hadamard's theorem remains valid if the positive curvature is replaced by the nonnegative curvature.

For $n \geq 2$, the statement that a hypersurface with nonnegative sectional curvature whose second fundamental form is semi-positive definite was completely proved by Sacksteder [17], combined with the earlier results of van Heijenoort [9] and Hartman–Nirenberg [8]. A simpler proof was later provided by do Carmo–Lima [6]. The further study of convex hypersurfaces can be found in, for example, H. Wu [24] and the references therein.

Among various notions of the intrinsic curvature, the sectional curvature is the strongest (pointwise) curvature condition, while the scalar curvature is the weakest. In this paper, we consider the condition only on the scalar curvature without imposing any condition on the sectional curvature. We would like to know what kind of convexity can be implied by the nonnegative scalar curvature.

Our another motivation comes from the study of the k -th mean curvature. For an n -dimensional hypersurface, its k -th mean curvature ($1 \leq k \leq n$), denoted by σ_k , is defined to be the k -th symmetric polynomial of its principal curvatures. It is known that σ_{2k} is intrinsically defined, while σ_{2k-1} is not, for each k (see [16], for example). In particular, σ_1 is the mean curvature, $2\sigma_2$ is the scalar curvature, and σ_n is the Gauss–Kronecker curvature.

If a closed smooth hypersurface has positive k -th mean curvature, then its l -th mean curvature is positive for each $1 \leq l \leq k$ (see, for example, [12, Proof of Proposition 3.3] and [4]). In particular, when $k = 2$, the result follows from the Gauss equation. A conjecture proposed by Huisken and Sinestrari [12, Proposition 3.3] asks whether the analogous result holds when one replaces the condition $\sigma_k > 0$ by $\sigma_k \geq 0$. However, this conjecture turns out not true, for all $k \geq 3$. We construct in Section 4 a family of examples with $\sigma_k \geq 0$ but $\sigma_1 < 0$ somewhere, for $k \geq 3$. These examples are inspired by Chern–Lashof [5].

In contrast to the counter-examples for all $k \geq 3$, the conjecture holds for $k = 2$. More precisely, we have the following result. Throughout this article, we assume that hypersurfaces are embedded and orientable.

Theorem 1. *Let $n \geq 2$ and M a closed embedded n -dimensional C^{n+1} hypersurface in \mathbb{R}^{n+1} . If the scalar curvature of M is nonnegative, then the mean curvature H has a sign, i.e., either $H \geq 0$ or $H \leq 0$ everywhere on M .*

The main analytic difficulty is that several natural geometric differential equations, including the linearized scalar curvature equation and scalar curvature flow, may be fully degenerate at points of vanishing mean curvature and cease to be globally elliptic. A new ingredient in the proof of Theorem 1 is that we consider the level sets of the height function on the hypersurface. We derive a geometric inequality which relates the mean curvature and scalar curvature of M to the mean curvature of the level sets (see Theorem 2.2). Therefore, the geometry of M has some quantitative influence on the geometry of its level sets. This inequality is rather general and holds at each point. We then carefully investigate the geometry of the level sets, and establish a key lemma (Lemma 3.5). This enables us to obtain the following more general result for a complete hypersurface (not necessarily closed), which may have interests of its own.

Theorem 2. *Let $n \geq 2$ and M a complete embedded n -dimensional C^{n+1} hypersurface in \mathbb{R}^{n+1} with nonnegative scalar curvature. Let H be the mean curvature of M . If there exists a connected component of $\{p \in M : H(p) = 0\}$ separating the subsets $\{p \in M : H(p) > 0\}$ and $\{p \in M : H(p) < 0\}$ in M , then the component must be unbounded.*

As an application of Theorem 1, we show that nonnegative scalar curvature is preserved by the mean curvature flow. More precisely, we have the following result.

Theorem 3. *Let $n \geq 2$ and M a closed embedded n -dimensional C^{n+1} hypersurface in \mathbb{R}^{n+1} with nonnegative scalar curvature. Let $\{M_t\}$ be a solution to the mean curvature flow with initial hypersurface M . Then, the scalar curvature of M_t is strictly positive for all $t > 0$.*

Moreover, by using Theorem 2, we can provide a simple proof to Sacksteder's theorem for the case of closed hypersurfaces (see Theorem 3.10). We expect the geometric inequality to have further applications. For example, in a forthcoming work [11] we shall prove some rigidity results for hypersurfaces in sphere, parallel to our previous work [10] in non-positive space form. (We refer readers to the excellent survey by Brendle [3] and the references therein, for the recent rigidity results involving scalar curvature.)

For a general non-closed hypersurface with nonnegative scalar curvature, the mean curvature may change signs locally. For example, consider the n -dimensional graph in \mathbb{R}^{n+1} defined by the function $f(x^1, \dots, x^n) = (x^n)^3$ over the unit ball centered at the origin in \mathbb{R}^n . The scalar curvature of the graph is zero, but its mean curvature is strictly positive when $x^n > 0$ and strictly negative when $x^n < 0$. Theorem 1 implies that the piece of hypersurface cannot be extended to a closed hypersurface with nonnegative scalar curvature.

By extending the argument in the proof of Theorem 2, we generalize Theorem 1 to complete non-compact hypersurfaces of finitely many regular ends (see Definition 5.1).

Theorem 4. *Let $n \geq 2$ and M a complete embedded n -dimensional C^{n+1} hypersurface in \mathbb{R}^{n+1} of finitely many regular ends with scalar curvature $R \geq 0$. Then the mean curvature H of M has a sign, i.e. either $H \geq 0$ or $H \leq 0$ on M .*

Furthermore, we prove a positive mass theorem for such hypersurfaces for all $n \geq 2$. For $n = 3$ and for asymptotically flat manifolds with nonnegative scalar curvature, the positive mass theorem was proved by Schoen–Yau [20, 21] and Witten [23]. The proofs have been generalized to asymptotically flat manifolds of dimension $3 \leq n \leq 7$ or to spin manifolds of dimension $n \geq 3$. For higher dimensional non-spin manifolds, some approaches have been announced by Lockamp [14] and by Schoen [19], which involve very delicate analysis. Recently, Lam [13] proved the positive mass inequality for graphical asymptotically flat hypersurfaces for all $n \geq 2$. See Bray [2] for a thorough and up-to-date survey article on Riemannian positive mass theorem. Our result is as follows:

Theorem 5. *Let $n \geq 2$ and M a complete embedded n -dimensional C^{n+1} hypersurface in \mathbb{R}^{n+1} of finitely many regular ends with nonnegative scalar curvature. Then, the mass on each end is nonnegative. Moreover, if M is connected and the mass of one end is zero, then M is identical to a hyperplane.*

Notice that in Theorem 5 we impose *no* decay condition on the induced metric of the ends. Generally, the mass defined by (5.3) may be $+\infty$. In Lemma 5.8, we show that the mass is finite and coincides with the classical definition of the ADM mass if the growth rate of the end is controlled. Our conditions are rather general and include interesting examples such as n -dimensional Schwarzschild manifolds embedded in \mathbb{R}^{n+1} with two ends (see Example 5.2). We remark that although hypersurfaces in \mathbb{R}^{n+1} are spin, Theorem 5 holds under more general asymptotics and does not seem to be a special case of the positive mass theorem for spin manifolds.

The article is organized as follows. In Section 2, we prove the geometric inequality (Theorem 2.2). In Section 3, we prove Theorem 1 and Theorem 2, and apply the results to study the mean curvature flow. In addition, we give a simple proof to Sacksteder's theorem for closed hypersurfaces. In Section 4, we provide the examples of non-mean convex hypersurfaces satisfying $\sigma_k \geq 0$, for all $k \geq 3$. Finally, Theorem 4 and Theorem 5 are proven in Section 5.

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2. THE MEAN CURVATURE OF THE LEVEL SETS

Let us begin with a linear algebra identity, which applies to a real matrix not necessarily being symmetric.

Proposition 2.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix with $n \geq 2$. Denote*

$$\sigma_1(A) = \sum_{i=1}^n a_{ii}, \quad \sigma_1(A|1) = \sum_{i=2}^n a_{ii}, \quad \sigma_2(A) = \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji}).$$

Then, we have

$$(2.1) \quad \begin{aligned} \sigma_1(A)\sigma_1(A|1) &= \sigma_2(A) + \frac{n}{2(n-1)}[\sigma_1(A|1)]^2 + \sum_{1 \leq i < j \leq n} a_{ij}a_{ji} \\ &\quad + \frac{1}{2(n-1)} \sum_{2 \leq i < j \leq n} (a_{ii} - a_{jj})^2, \end{aligned}$$

where the last term is zero when $n = 2$. In particular, if A is real and $a_{ij}a_{ji} \geq 0$ for all $1 \leq i < j \leq n$, then

$$\sigma_1(A)\sigma_1(A|1) \geq \sigma_2(A) + \frac{n}{2(n-1)}[\sigma_1(A|1)]^2,$$

where “=” holds if and only if $a_{22} = \cdots = a_{nn}$ and $a_{ij}a_{ji} = 0$ for all $i, j = 1, \dots, n$ and $i \neq j$.

Proof. Note that

$$\sigma_2(A) = a_{11}\sigma_1(A|1) + \sum_{2 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}.$$

Then,

$$\begin{aligned} \sigma_1(A)\sigma_1(A|1) &= a_{11}\sigma_1(A|1) + \sigma_1(A|1)^2 \\ (2.2) \quad &= \sigma_2(A) + \sum_{1 \leq i < j \leq n} a_{ij}a_{ji} + \sigma_1(A|1)^2 - \sum_{2 \leq i < j \leq n} a_{ii}a_{jj}. \end{aligned}$$

Now (2.1) follows from applying

$$\frac{n-2}{2(n-1)} \left(\sum_{j=2}^n a_{jj} \right)^2 - \sum_{2 \leq i < j \leq n} a_{ii}a_{jj} = \frac{1}{2(n-1)} \sum_{2 \leq i < j \leq n} (a_{ii} - a_{jj})^2$$

to the last term on the right hand side of (2.2). \square

Let M be a complete C^2 hypersurface in \mathbb{R}^{n+1} . Assume that M is embedded and orientable. Consider the height function $h : M \rightarrow \mathbb{R}$ given by

$$h(x^1, \dots, x^{n+1}) = x^{n+1}.$$

By Morse–Sard theorem (see [18] for example), for almost every height value, the level set of h is an $(n-1)$ -dimensional C^2 manifold in \mathbb{R}^n . Without loss of generality, we assume that 0 is a regular value of h , i.e.,

$$\Sigma = M \cap \{x^{n+1} = 0\}$$

is a C^2 hypersurface in \mathbb{R}^n , and $|\nabla^M h| \neq 0$ for every point in Σ .

We shall adopt the following convention for the mean curvature. Let N be a (piece) of hypersurface in Euclidean space. Let μ be a non-vanishing unit normal vector field to N . The mean curvature of N defined by μ is given by

$$H_N = -\operatorname{div}_0 \mu,$$

where div_0 is the Euclidean divergence operator. The n -dimensional sphere of radius r has positive mean curvature n/r with respect to the inward unit normal vector by this convention.

Moreover, we denote by $\langle \cdot, \cdot \rangle$ the standard metric on \mathbb{R}^{n+1} , and denote by $\partial_1, \dots, \partial_{n+1}$ the tangent vectors with respect to $(\mathbb{R}^{n+1}; x^1, \dots, x^{n+1})$. For a C^2 function f , we abbreviate $f_i = \partial f / \partial x^i$, $f_{ij} = \partial^2 f / \partial x^i \partial x^j$, and denote $Df = (f_1, \dots, f_n)$.

Theorem 2.2. *Let M and h be given as before, and $\Sigma = h^{-1}(0)$ with $|\nabla^M h| > 0$ on Σ . Denote by ν and η the unit normal vector fields to $M \subset \mathbb{R}^{n+1}$ and $\Sigma \subset \mathbb{R}^n$, respectively; and denote by H and H_Σ the mean curvatures of $M \subset \mathbb{R}^{n+1}$ and $\Sigma \subset \mathbb{R}^n$ defined by ν and η , respectively. Let R be the induced scalar curvature of M . Then,*

$$(2.3) \quad \langle \nu, \eta \rangle H H_\Sigma \geq \frac{R}{2} + \frac{n}{2(n-1)} \langle \nu, \eta \rangle^2 H_\Sigma^2 \quad \text{on } \Sigma,$$

where “=” holds at a point in Σ if and only if (M, Σ) satisfies the following two conditions at the point:

- (i) $\Sigma \subset \mathbb{R}^n$ is umbilic, with the principal curvature κ ;
- (ii) $M \subset \mathbb{R}^{n+1}$ has at most two distinct principal curvatures, and one of them is equal to $\langle \nu, \eta \rangle \kappa$, with multiplicity at least $n - 1$.

Proof. It suffices to show (2.3) at a point $p \in \Sigma$. We may assume $\langle \nu, \eta \rangle \geq 0$ at p . Otherwise, we can replace η by $-\eta$. Let us divide the proof into two cases:

Case 1: Assume that $\langle \nu, \eta \rangle < 1$ at p . Since $\Sigma = M \cap \{x^{n+1} = 0\}$, we have in this case that $\langle \nu, \partial_{n+1} \rangle \neq 0$ at p . Then, a neighborhood V of p in M can be represented by

$$x^{n+1} = f(x), \quad \text{for all } x = (x^1, \dots, x^n) \in \Omega,$$

in which $\Omega \subset \{x^{n+1} = 0\}$ is a small domain containing p , and $f \in C^2(\Omega)$. It follows that

$$(2.4) \quad \Sigma \cap V = \{x \in \Omega \mid f(x) = 0\}.$$

We assume, without loss of generality, that $\langle \nu, \partial_{n+1} \rangle > 0$ at p ; then

$$(2.5) \quad \nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}} \quad \text{at } p.$$

We remark that $|Df| \equiv \sqrt{f_1^2 + \dots + f_n^2} > 0$ at p , for, by the construction we have

$$h(x, f(x)) = f(x) \quad \text{for all } x \in \Omega;$$

thus, $|\nabla^M h| > 0$ on Σ implies that $|Df| > 0$ on $\Sigma \cap V$. We can rotate the coordinates (x^1, \dots, x^n) in Ω so that at p

$$f_1 = |Df|, \quad \text{and} \quad f_\alpha = 0 \quad \text{for all } 2 \leq \alpha \leq n.$$

Then, the shape operator $A = (A_j^i)$ on $M \subset \mathbb{R}^{n+1}$ at p is given by

$$A_j^i = \frac{\partial}{\partial x^j} \left(\frac{f_i}{w} \right) = \frac{1}{w} \left(f_{ij} - \frac{\delta_{i1} f_{1j} |Df|^2}{w^2} \right),$$

where $w \equiv \sqrt{1 + |Df|^2}$. Hence, at the point p ,

$$(2.6) \quad \begin{aligned} \sigma_1(A) &= H = \frac{f_{11}}{w^3} + \frac{1}{w} \sum_{\alpha \geq 2} f_{\alpha\alpha}, \\ \sigma_1(A|1) &= \sum_{\alpha \geq 2} A_\alpha^\alpha = \frac{1}{w} \sum_{\alpha \geq 2} f_{\alpha\alpha}. \end{aligned}$$

On the other hand, by (2.4) and (2.5) we have $\eta = -Df/|Df|$ at $p \in \Sigma$; hence,

$$\langle \nu, \eta \rangle = \frac{|Df|}{w} > 0 \quad \text{at } p.$$

Furthermore, the shape operator A_Σ on $\Sigma \subset \mathbb{R}^n$ is given by

$$(2.7) \quad (A_\Sigma)_j^i = \frac{\partial}{\partial x^j} \left(\frac{f_i}{|Df|} \right) = \frac{f_{ij}}{|Df|}, \quad 2 \leq i, j \leq n.$$

In particular, the mean curvature

$$(2.8) \quad H_\Sigma = \sum_{i=2}^n (A_\Sigma)_i^i = \frac{1}{|Df|} \sum_{\alpha=2}^n f_{\alpha\alpha}.$$

Comparing (2.6) and (2.8) we have

$$\sigma_1(A|1) = \frac{|Df|}{w} H_\Sigma = \langle \nu, \eta \rangle H_\Sigma.$$

Now applying Proposition 2.1 with $\sigma_2(A) = R/2$ yields

$$\langle \nu, \eta \rangle H H_\Sigma \geq \frac{R}{2} + \frac{n}{2(n-1)} (\langle \nu, \eta \rangle H_\Sigma)^2.$$

Here the equality holds if and only if

$$f_{22} = \cdots = f_{nn}, \quad \text{and} \quad f_{ij} = 0 \quad \text{for all } i \neq j,$$

which, by (2.7), is the same as that (M, Σ) satisfies conditions (i) and (ii) at p . This proves the result for Case 1.

Case 2: Assume that $\langle \nu, \eta \rangle = 1$ at p . Then, $\nu = \eta$ at p ; equivalently,

$$\langle \nu, \partial_{n+1} \rangle = 0.$$

Let us assume, without loss of generality, that $\langle \nu, \partial_1 \rangle \neq 0$. We can furthermore rotate the coordinates (x^1, \dots, x^n) so that

$$\nu = \partial_1 \quad \text{at } p.$$

Then, by the implicit function theorem, we can represent a neighborhood U of p in M by

$$x^1 = \psi(x^2, \dots, x^n, x^{n+1}), \quad \text{for all } (x^2, \dots, x^{n+1}) \in \Omega_1,$$

where $\Omega_1 \subset \{x^1 = 0\}$ is a small domain containing p , and $\psi \in C^2(\Omega_1)$ satisfies that

$$(2.9) \quad \psi_i(p) = 0 \quad \text{for all } 2 \leq i \leq n+1.$$

Since $\Sigma \subset \{x^{n+1} = 0\}$, $\Sigma \cap U$ is given by

$$x^1 = \psi(x', 0) \quad \text{for all } (x', 0) \equiv (x^2, \dots, x^n, 0) \in \Omega_1.$$

By construction above, we have

$$\nu = \frac{(1, -D'\psi, -\psi_{n+1})}{\sqrt{1 + |D'\psi|^2 + \psi_{n+1}^2}}, \quad \text{and} \quad \eta = \frac{(1, -D'\psi, 0)}{\sqrt{1 + |D'\psi|^2}},$$

where $D'\psi = (\psi_2, \dots, \psi_n)$. Using (2.9) we obtain the shape operator A for $M \subset \mathbb{R}^{n+1}$ at p

$$A_j^i = \frac{\partial}{\partial x^j} \left(\frac{\psi_i}{\sqrt{1 + |D'\psi|^2 + \psi_{n+1}^2}} \right) = \psi_{ij}, \quad 2 \leq i, j \leq n+1,$$

while the shape operator A_Σ for $\Sigma \subset \mathbb{R}^n$ at p is

$$(2.10) \quad (A_\Sigma)_j^i = \frac{\partial}{\partial x^j} \left(\frac{\psi_i}{\sqrt{1 + |D'\psi|^2}} \right) = \psi_{ij}, \quad \text{for all } 2 \leq i, j \leq n.$$

Hence, at p the matrix A_Σ is exactly the first $(n-1) \times (n-1)$ principal minor of the matrix A . It then follows from Proposition 2.1 that

$$H_\Sigma H \geq \frac{R}{2} + \frac{n}{2(n-1)} H_\Sigma^2,$$

where “=” holds if and only if

$$\psi_{22} = \dots = \psi_{nn}, \quad \text{and} \quad \psi_{ij} = 0 \text{ for all } i \neq j,$$

which is the same as that (M, Σ) satisfies conditions (i) and (ii) at p , by (2.10). This proves the result for Case 2. Combining the two cases, we finish the proof. \square

Corollary 2.3. *With the notations in Theorem 2.2, if $R \geq 0$ on M , then*

$$\langle \nu, \eta \rangle H H_\Sigma \geq \frac{n}{2(n-1)} \langle \nu, \eta \rangle^2 H_\Sigma^2 \quad \text{on } \Sigma.$$

In particular, $\langle \nu, \eta \rangle H H_\Sigma \geq 0$ at the point; in addition, if $H = 0$, then $H_\Sigma = 0$, and both M and Σ are geodesic at the point.

3. COMPLETE HYPERSURFACES WITH NONNEGATIVE SCALAR CURVATURE

Let us first prove some results on the mean curvature operator. Let f be a C^2 -function defined over an open set in \mathbb{R}^n . The upward unit normal vector of the graph of f is

$$(3.1) \quad \nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.$$

The mean curvature operator is defined by

$$H(f) := -\operatorname{div}_0 \nu = \sum_{i,j=1}^n \left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) \frac{f_{ij}}{\sqrt{1 + |Df|^2}}.$$

Proposition 3.1. *Let W be an open subset in \mathbb{R}^n , not necessarily bounded. Let $p \in \partial W$, and denote by $B(p)$ a small open ball in \mathbb{R}^n centered at p . Suppose that $f \in C^2(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ satisfies*

$$\begin{aligned} H(f) &\geq 0 && \text{in } W \cap B(p) \\ f = c, \quad |Df| &= 0 && \text{on } \partial W \cap B(p), \end{aligned}$$

for some constant c . Then either $f \equiv c$ in $W \cap B(p)$, or

$$\{x \in W \cap B(p) : f(x) > c\} \neq \emptyset.$$

Remark 3.2. Notice that we impose no hypothesis on regularity of ∂W here and below.

Proof. Without loss of generality, we can assume $c = 0$. We prove by contradiction. Suppose that $f \leq 0$ and f is not identically zero on $W \cap B(p)$. If $f = 0$ at $x_0 \in W \cap B(p)$, then x_0 is a local maximum of f . This contradicts the strong maximum principle applied to $H(f) \geq 0$. Thus, f must be strictly negative in $W \cap B(p)$. Because $W \cap B(p)$ is open, we can take a smaller open ball B contained in $W \cap B(p)$ such that ∂B touches $\partial W \cap B(p)$ at a point q ; in other words, ∂W satisfies an interior sphere condition at $q \in \partial W$. Furthermore, $f < 0$ on B and $f(q) = 0$. Applying the Hopf boundary lemma to the mean curvature operator $H(f)$ yields that $|Df|(q) \neq 0$, which contradicts with the assumption that $|Df|(q) = 0$ for $q \in \partial W \cap B(p)$. \square

Definition 3.3. Let W be a subset in \mathbb{R}^n . A point $p \in \partial W$ is called a *convex point* of W , if there exists an $(n-1)$ -sphere S in \mathbb{R}^n passing through p so that $\overline{W} \setminus \{p\}$ is contained in the open ball enclosed by S . Here \overline{W} denotes the closure of W in \mathbb{R}^n .

Remark 3.4. A bounded subset in \mathbb{R}^n has at least one convex point.

Lemma 3.5. Let W be an open subset in \mathbb{R}^n , not necessarily bounded. Suppose that $p \in \partial W$ is a convex point of W . Denote by $B(p)$ an open ball in \mathbb{R}^n centered at p . Suppose that $f \in C^2(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ and $f = c, |Df| = 0$ on $\partial W \cap B(p)$ for some constant c . If the scalar curvature of the graph of f is nonnegative, then $H(f)$ must change signs in $W \cap B(p)$, unless $f \equiv c$ on $W \cap B(p)$.

Proof. It suffices to prove the lemma for $c = 0$. Suppose on the contrary that $H(f)$ does not change signs over $W \cap B(p)$. Then, we may assume that $H(f) \geq 0$ over $W \cap B(p)$, for, otherwise, we can replace f by $-f$.

Because p is a convex point, there exists an $(n-1)$ -sphere S through p so that $\overline{W} \setminus \{p\}$ is contained in the open ball enclosed by S . Consider the level sets of f inside $W \cap B(p)$

$$\Sigma_\epsilon = \{x \in W \cap B(p) : f(x) = \epsilon\}.$$

By Proposition 3.1, Σ_ϵ is non-empty for all $\epsilon > 0$ small enough. By Morse–Sard theorem, for almost every ϵ , Σ_ϵ is a piece of C^2 hypersurface in \mathbb{R}^n . Note that either Σ_ϵ is closed, or $\partial \Sigma_\epsilon$ is nonempty and is contained in $W \cap \partial B(p)$.

We pick a sufficiently small ϵ so that $|Df|$ never vanishes on Σ_ϵ , and the distance from the level set Σ_ϵ to the $(n-1)$ -sphere S is less than the distance from $\overline{W} \cap \partial B(p)$ to S . This can be done because p is a convex point of W . Now we continuously translate the $(n-1)$ -sphere S toward Σ_ϵ along its inward normal at p , until ultimately it begins to intersect Σ_ϵ . Denote

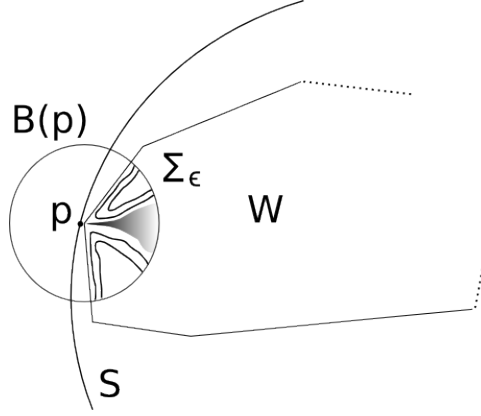


FIGURE 1. As in the proof of Lemma 3.5, W is an open subset in \mathbb{R}^n , p is a convex point of W , and S is an $(n-1)$ -sphere through p . For any small ball $B(p)$ centered at p , $\{x \in W \cap B(p) : f(x) > c\} \neq \emptyset$, unless $f \equiv c$. Hence the level set $\Sigma_{c+\epsilon}$ of f is nonempty for $\epsilon > 0$ small. The shaded region represents the set $\{x \in W \cap B(p) : f < c\}$ (possibly empty).

by S' the resulting $(n-1)$ -sphere. Then, S' must be tangent to Σ_ϵ at an interior point x_0 , and Σ_ϵ lies in the ball enclosed by S' . Let $\eta = Df/|Df|$ be the normal vector to Σ_ϵ in \mathbb{R}^n and H_{Σ_ϵ} the mean curvature of Σ_ϵ with respect to η . Note that η at x_0 is pointing inward to W , because $f = 0$ on $\partial W \cap B(p)$ and x_0 is the intersection of Σ_ϵ and the $(n-1)$ -sphere S' for the first time. By comparison principle, $H_{\Sigma_\epsilon} > 0$ at the tangent point x_0 . On the other hand, since the scalar curvature of the graph of f is nonnegative, it follows from Corollary 2.3 that

$$\langle \nu, \eta \rangle H H_{\Sigma_\epsilon} \geq 0.$$

By (3.1), $\langle \nu, \eta \rangle < 0$ on Σ_ϵ . Therefore, $H_{\Sigma_\epsilon} \leq 0$ at x_0 . This leads to a contradiction. \square

Lemma 3.6. *Denote by B_r the open ball in \mathbb{R}^n centered at the origin of radius r . Let $f \in C^2(B_{r_2} \setminus \overline{B_{r_1}}) \cap C^1(\overline{B_{r_2}} \setminus \overline{B_{r_1}})$ for some $r_2 > r_1 > 0$. Suppose that f satisfies $H(f) \geq 0$ and the scalar curvature of the graph of f is nonnegative. Then*

$$\max_{\overline{B_{r_2}} \setminus B_{r_1}} f = \max_{\partial B_{r_2}} f.$$

Moreover, if $f(x) = \max_{\partial B_{r_2}} f$ for some interior point $x \in B_{r_2} \setminus \overline{B_{r_1}}$, then

$$f \equiv \sup_{\partial B_{r_2}} f \text{ in } \overline{B_{r_2}} \setminus \overline{B_{r_1}}.$$

Remark 3.7. Lemma 3.6 does not follow directly from applying the standard maximum principle to $H(f) \geq 0$, since we impose no hypothesis on $\max_{\partial B_{r_1}} f$. This result may have interests of its own.

Proof. By subtracting $\max_{\partial B_{r_2}} f$ from f , we may assume $\max_{\partial B_{r_2}} f = 0$. Suppose on the contrary that f is not identically zero and $f > 0$ somewhere in $B_{r_2} \setminus B_{r_1}$. Then the level set

$$\Sigma_\epsilon = \{x \in B_{r_2} \setminus B_{r_1} : f(x) = \epsilon\}$$

is non-empty for $\epsilon > 0$ sufficiently small. By Morse–Sard theorem, for almost every small ϵ , Σ_ϵ is a piece of C^2 hypersurface in \mathbb{R}^n . Note that either Σ_ϵ has no boundary, or $\partial\Sigma_\epsilon$ is contained in ∂B_{r_1} . Fix $\epsilon > 0$ so that $|Df|$ does not vanish on Σ_ϵ . Let $p \in \partial B_{r_2}$ be a point that is closest to Σ_ϵ . Now we continuously translate the $(n-1)$ -sphere ∂B_{r_2} toward Σ_ϵ , along its inward normal at p . Denote by S' the $(n-1)$ -sphere that touches Σ_ϵ for the first time. Then, S' must be tangent to Σ_ϵ at an interior point x_0 , and Σ_ϵ lies in the ball enclosed by S' . The rest of proof is similar to the proof of Lemma 3.5, so we only highlight the key argument. By comparison principle, the mean curvature H_{Σ_ϵ} of Σ_ϵ with respect to the inward unit normal vector $Df/|Df|$ is positive at x_0 . However, it contradicts $H_{\Sigma_\epsilon} \leq 0$ at x_0 from Corollary 2.3.

Moreover, if $f(x) = \max_{\partial B_{r_2}} f$ for some interior point x , then by strong maximum principle and $H(f) \geq 0$ we prove the second statement. \square

In the rest of this section, we let M be a complete, embedded and orientable C^{n+1} -smooth hypersurface in \mathbb{R}^{n+1} , unless otherwise indicated. We denote by R and H the scalar curvature and mean curvature of M , respectively, and denote by A the second fundamental form of M . We also abbreviate $\{H > 0\} = \{q \in M : H(q) > 0\}$ and $\{H < 0\} = \{q \in M : H(q) < 0\}$.

Let M_0 be the set of geodesic points in M , i.e.,

$$M_0 = \{q \in M : A(q) = 0\}.$$

Suppose that M has nonnegative scalar curvature. Then, by Gauss equation,

$$H^2 - |A|^2 = R \geq 0$$

implies that a point in M with $H = 0$ is also a geodesic point; that is,

$$M_0 = \{q \in M : H(q) = 0\}.$$

We recall the following characterization of the set of geodesic points, due to Sacksteder [17]. For the sake of completeness, we include his proof. This is the only place that requires the hypersurface M to be C^{n+1} , instead of C^2 , in our main theorems (Theorems 1 to 5). Notice that the assumption that $R \geq 0$ is not needed in the proof of Lemma 3.8.

Lemma 3.8. *Suppose that M is a C^{n+1} hypersurface in \mathbb{R}^{n+1} . Let M'_0 be a connected component of M_0 . Then M'_0 lies in a hyperplane which is tangent to M at every point in M'_0 .*

Proof. We consider the Gauss map $\nu : M \rightarrow \mathbb{S}^n$. Since M is of C^{n+1} , the Gauss map is of C^n . Note that the Gauss map ν has rank zero at any geodesic point. We can then apply a theorem of Sard [18, p. 888, Theorem

6.1] to obtain that the image $\nu(M_0)$ has Hausdorff dimension zero in \mathbb{S}^n . It follows that $\nu(M_0)$ is totally disconnected in \mathbb{S}^n . Thus, $\nu(M'_0)$ consists of a single point in \mathbb{S}^n , denoted by ν_0 .

It remains to show that M'_0 lies in a hyperplane which is orthogonal to ν_0 . Pick a point $p_0 \in M'_0$. Let $(V; y^1, \dots, y^n)$ be a local coordinate chart centered at p_0 in M , and define

$$\varphi(y) = \langle \nu_0, x(y) - x(0) \rangle, \quad \text{for each } y = (y^1, \dots, y^n) \in V.$$

Here $x = (x^1, \dots, x^{n+1})$ is the coordinates in \mathbb{R}^{n+1} , and $\langle \cdot, \cdot \rangle$ is the Euclidean metric on \mathbb{R}^{n+1} . Then, the function $\varphi \in C^{n+1}(V)$, and by our construction,

$$M'_0 \cap V \subset \{y \in V \mid \frac{\partial \varphi}{\partial y^i}(y) = 0, i = 1, \dots, n\}.$$

It follows from a theorem of A. P. Morse [15, p. 70, Theorem 4.4] that φ is a constant on $M'_0 \cap V$; thus, $\varphi \equiv 0$ on $M'_0 \cap V$. Since M'_0 is connected,

$$\langle \nu_0, x(p) - x(p_0) \rangle = 0 \quad \text{for all } p \in M'_0,$$

namely, M'_0 lies in the hyperplane orthogonal to ν_0 . \square

Definition 3.9. Let N be a complete manifold and N_1 and N_2 be two nonempty disconnected open subsets of N . For a closed subset E in N , we say that E separates N_1 and N_2 in N , if there exists an open neighborhood V of E in N so that $V \setminus E = N_1 \cup N_2$.

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Suppose on the contrary that M'_0 is a bounded connected component of M_0 and that M'_0 separates the subsets $\{H > 0\}$ and $\{H < 0\}$ in M . By Lemma 3.8, M'_0 lies in a tangent plane to M . Assume that the tangent plane is $\{x^{n+1} = 0\}$. Then, near M'_0 the hypersurface M can be represented as the graph of a C^{n+1} -function f in a neighborhood of M'_0 in $\{x^{n+1} = 0\}$, which satisfies $f = 0, |Df| = 0$ on M'_0 . Because M is complete and M'_0 separates $\{H > 0\}$ and $\{H < 0\}$ in M , M'_0 also separates regions in $\{x^{n+1} = 0\}$. We denote by W one of the connected open subsets in $\{x^{n+1} = 0\}$ enclosed by M'_0 . Then, $\partial W \subset M'_0$, and W has a convex point $p \in \partial W$. For a small open ball $B(p)$, we have either $H(f) > 0$, or $H(f) < 0$ in $W \cap B(p)$. However, it contradicts Lemma 3.5. \square

The proof of Theorem 1 follows directly from Theorem 2.

Proof of Theorem 1. Because M has nonnegative scalar curvature, the set of points where $H = 0$ is equal to M_0 , the set of geodesic points. Suppose on the contrary that H changes signs. Then, a connected component of M_0 would separate the sets $\{H > 0\}$ and $\{H < 0\}$. The component is bounded because M is closed. This contradicts Theorem 2. \square

Using Theorem 2, we provide another proof to Sacksteder's theorem for closed hypersurfaces.

Theorem 3.10. *Suppose that M is a C^{n+1} -smooth closed hypersurface in \mathbb{R}^{n+1} . If the sectional curvature of M is nonnegative, then the induced second fundamental form is semi-positive definite. As a consequence, M is the boundary of a convex body in \mathbb{R}^{n+1} .*

Proof. Let (A_{ij}) be the second fundamental form of M . Then, at a point in M , the principal curvatures are either all nonnegative or all non-positive, because the sectional curvature of M is nonnegative. Suppose on the contrary that (A_{ij}) is not semi-positive definite. Then, there exists a connected component M'_0 of M_0 , which separates the sets $\{(A_{ij}) > 0\}$ and $\{(A_{ij}) < 0\}$ in M . In particular, M'_0 separates the sets $\{H > 0\}$ and $\{H < 0\}$ in M . Since M is compact, M'_0 is necessarily bounded. This contradicts with Theorem 2. \square

Theorem 1 can be applied to study the mean curvature flow, especially when the initial hypersurface is closed and has nonnegative scalar curvature. Let us briefly recall the setting of the mean curvature flow, and then prove Theorem 3. Let M be a closed hypersurface in \mathbb{R}^{n+1} represented by a diffeomorphism: For an open subset $U \subset \mathbb{R}^n$,

$$F_0 : U \rightarrow F_0(U) \subset M \subset \mathbb{R}^{n+1}.$$

Let $F(x, t)$ be a family of maps satisfying

$$\begin{aligned} \frac{\partial}{\partial t} F(x, t) &= H(x, t) \nu(x, t), \quad x \in U, \\ F(\cdot, 0) &= F_0, \end{aligned}$$

where $\nu(\cdot, t)$ is the inward unit normal vector to $M_t := F(M, t)$ and $H(\cdot, t)$ is the mean curvature with respect to ν . The family of closed hypersurfaces $\{M_t\}$ for $t > 0$ is called a solution to the mean curvature flow.

Proof of Theorem 3. By Theorem 1, the mean curvature of M is nonnegative. It is well known that if M has nonnegative mean curvature, then M_t has positive mean curvature for all $t > 0$. We then consider

$$q_2 := \frac{R}{2H},$$

where H and R are the mean curvature and scalar curvature of M_t , respectively. A result of Huisken–Sinestrari [12, p. 61, Corollary 3.2] shows that the evolution equation of q_2 satisfies the parabolic strong maximum principle. It follows that $q_2 > 0$ on M_t for all $t > 0$, because $q_2 \geq 0$ on M . Thus, we conclude that $R > 0$ on M_t for all $t > 0$. \square

4. EXAMPLES OF NONNEGATIVE k -TH MEAN CURVATURE

Let M be a smooth closed hypersurface in \mathbb{R}^{n+1} . Denote by κ_i , $i = 1, \dots, n$, the principal curvatures of M . We define, for each $1 \leq k \leq n$, the k -th mean curvature of M to be

$$\sigma_k(A) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

In particular, $\sigma_1(A)$, $2\sigma_2(A)$, and $\sigma_n(A)$ are the mean curvature, the scalar curvature, and the Gauss–Kronecker curvature of M , respectively.

It is well known that if $\sigma_k(A) > 0$ then $\sigma_l(A) > 0$ for all $1 \leq l \leq k$ (see, for example, [4] and [12, p. 51]). We are interested in the non-strict inequality case: Namely, the question is, *whether $\sigma_k(A) \geq 0$ would imply $\sigma_l(A) \geq 0$ for all $1 \leq l \leq k$?*

For $k = 2$, Theorem 1 tells us that $\sigma_2(A) \geq 0$ implies $\sigma_1(A) \geq 0$. However, it is no longer true for $k \geq 3$. In fact, for any $n \geq 3$ and $k \geq 3$, we construct below a family of smooth closed hypersurfaces, which satisfy $\sigma_k(A) \geq 0$ but are not mean convex, i.e., $\sigma_1(A)$ changes signs.

Example 4.1. Let $n \geq 3$ and k be an odd integer such that $3 \leq k \leq n$. Consider the hypersurface in \mathbb{R}^{n+1} given by

$$(4.1) \quad (r - a)^2 + (x^{n+1})^2 = 1,$$

where $a > 1$ is a constant, and

$$r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

This hypersurface is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, for it can be obtained by rotating a unit circle about the x^{n+1} -axis.

We shall show that, for each $a \in (n/k, n)$, $\sigma_k(A) \geq 0$ but $\sigma_1(A)$ change signs. (When $k < n$, the same conclusion holds if $a = n/k$.) In particular, letting $a = n/2$, the hypersurface given by (4.1) has nonnegative k -th mean curvature while its first mean curvature changes signs.

Note that the hypersurface is symmetric about $\{x^{n+1} = 0\}$. Let us consider the lower half portion $x^{n+1} = \phi(r)$, where

$$\phi(r) = -[1 - (r - a)^2]^{1/2}, \quad \text{for all } a - 1 \leq r \leq a + 1.$$

By a direct computation, the k -th mean curvature of the graph of $x^{n+1} = \phi(r)$ is

$$\begin{aligned} \sigma_k(A) &= \binom{n-1}{k-1} \frac{(\phi')^{k-1}}{r^{k-1}[1 + (\phi')^2]^{k/2}} \left(\frac{\phi''}{1 + (\phi')^2} + \frac{n-k}{k} \frac{\phi'}{r} \right) \\ &= \binom{n-1}{k-1} \left(1 - \frac{a}{r}\right)^{k-1} \left[1 + \frac{n-k}{k} \left(1 - \frac{a}{r}\right) \right]. \end{aligned}$$

Since k is odd, to get $\sigma_k(A) \geq 0$ it suffices to consider

$$0 \leq 1 + \frac{n-k}{k} \left(1 - \frac{a}{r}\right).$$

That is, when $1 \leq k < n$, $\sigma_k(A) \geq 0$ if

$$1 - \frac{a}{r} \geq -\frac{k}{n-k}, \quad \text{for all } a - 1 < r < a + 1;$$

and when $k = n$ (then n is odd), $\sigma_n(A)$ is always nonnegative. On the other hand,

$$\frac{1}{a+1} > 1 - \frac{a}{r} > -\frac{1}{a-1}, \quad \text{for } a - 1 < r < a + 1.$$

Therefore, for all the real numbers a satisfying

$$-\frac{1}{n-1} > -\frac{1}{a-1} > -\frac{k}{n-k}, \quad \text{i.e., } n > a > \frac{n}{k},$$

we have $\sigma_k(A) \geq 0$, but $\sigma_1(A)$ changes signs. More precisely, let us fix any real number $a \in (n/k, n)$; then

$$\begin{aligned} \sigma_1(A) &> 0, & \text{for } \frac{n-1}{n}a < r < a+1, & \text{and} \\ \sigma_1(A) &< 0, & \text{for } a-1 < r < \frac{n-1}{n}a. \end{aligned}$$

Example 4.2. Let $n \geq 4$, and k an even integer satisfying $4 \leq k \leq n$. We consider the smooth embedded hypersurface in \mathbb{R}^{n+1} given by the equation

$$(4.2) \quad (r-a)^2 + (x^n)^2 + (x^{n+1})^2 = 1,$$

where $a > 1$ is a constant to be determined. This hypersurface is obtained by rotating a 2-dimensional unit sphere about a 2-dimensional coordinate plane, and is therefore homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^{n-2}$.

We would like to prove that, for any $a \in (1+b(n,k), n/2)$ the hypersurface defined by (4.2) satisfies that $\sigma_k(A) \geq 0$ and $\sigma_1(A)$ changes signs, where

$$b(n,k) = \frac{n-k}{k-1} + \frac{1}{k-1} \sqrt{\frac{(n-1)(n-k)}{k}} \geq 0.$$

That the interval $(1+b(n,k), n/2)$ is nonempty for $n \geq k \geq 4$ is justified by Proposition 4.3 below. (However, the interval is empty for $k=2$ and all $n \geq 2$.)

We first derive a formula for $\sigma_k(A)$ for all $n \geq 3$ and $1 \leq k \leq n$. Let

$$\psi(r, x^n) = -\sqrt{1 - (r-a)^2 - (x^n)^2}, \quad 0 < a-1 \leq r \leq a+1.$$

By rotation symmetry, it suffices to carry out the calculation at a point where $x^1 = \dots = x^{n-2} = 0$ and $x^{n-1} = r$. Unless otherwise indicated, we let Greek letters such as α, β range from 1 to n , and English letters such as i, j range from 1 to $n-1$. We denote $\psi_\alpha = \partial\psi/\partial x^\alpha$ and $\psi_{\alpha\beta} = \partial^2\psi/\partial x^\alpha \partial x^\beta$.

Note that the induced metric is given by

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \psi_\alpha \psi_\beta.$$

It follows that the matrix, at the point $(0, \dots, 0, r, x^n)$,

$$(g_{\alpha\beta}) = \begin{bmatrix} I_{n-2} & 0 \\ 0 & T_2 \end{bmatrix}.$$

Here I_m denotes the $m \times m$ identity matrix, and T_2 is a 2×2 matrix defined by

$$\begin{aligned} (4.3) \quad T_2 &= \begin{bmatrix} 1 + (\psi_r)^2 & \psi_r \psi_n \\ \psi_r \psi_n & 1 + (\psi_n)^2 \end{bmatrix} \\ &= (-\psi)^{-2} \begin{bmatrix} 1 - (x^n)^2 & x^n(r-a) \\ x^n(r-a) & 1 - (r-a)^2 \end{bmatrix}, \end{aligned}$$

in which

$$\psi_r = \frac{\partial \psi}{\partial r} = \frac{r-a}{-\psi}, \quad \psi_n = \frac{\partial \psi}{\partial x^n} = \frac{x^n}{-\psi}.$$

On the other hand, the second fundamental form

$$A_{\alpha\beta} = \frac{\psi_{\alpha\beta}}{\sqrt{1+|D\psi|^2}} = (-\psi)\psi_{\alpha\beta}.$$

Then, at the point $(0, \dots, 0, r, x^n)$, the second fundamental form matrix

$$(A_{\alpha\beta}) = \begin{bmatrix} (1-a/r)I_{n-2} & 0 \\ 0 & T_2 \end{bmatrix},$$

where T_2 is the 2×2 matrix given by (4.3). Hence, the matrix of the shape operator is given by

$$(A_\alpha^\beta) = (A_{\alpha\gamma})(g_{\beta\gamma})^{-1} = \begin{bmatrix} (1-a/r)I_{n-2} & 0 \\ 0 & I_2 \end{bmatrix}.$$

Therefore, we have

$$(4.4) \quad H = \sigma_1(A) = (n-2)t + 2,$$

and for any $2 \leq k \leq n$,

$$(4.5) \quad \sigma_k(A) = \binom{n-2}{k} t^k + 2 \binom{n-2}{k-1} t^{k-1} + \binom{n-2}{k-2} t^{k-2}.$$

Here we denote $t = 1 - a/r$, which satisfies that

$$(4.6) \quad -\frac{1}{a-1} < t < \frac{1}{a+1}, \quad \text{for all } a-1 < r < a+1.$$

Moreover, in (4.5), we use the combinatoric convention so that

$$\sigma_n(A) = t^{n-2}, \quad \sigma_{n-1}(A) = 2t^{n-2} + (n-2)t^{n-3}.$$

Now return to our setting $n \geq k \geq 4$ and k being even. Clearly, for $k = n$ (thus n is even), we always have $\sigma_n(A) \geq 0$. For $k < n$ and k being even, $\sigma_k(A) \geq 0$ if

$$(4.7) \quad \binom{n-2}{k} t^2 + 2 \binom{n-2}{k-1} t + \binom{n-2}{k-2} \geq 0.$$

Note that the inequality (4.7) holds for all $t \geq t_1$, where

$$t_1 = -\frac{k-1}{n-k} \left[\sqrt{\frac{n-1}{k(n-k)}} + 1 \right]^{-1} = -\frac{1}{b(n,k)}.$$

On the other hand, by (4.4) we have $H < 0$ for

$$t < -\frac{2}{n-2}.$$

Thus, if we can show that $t_1 < -2/(n-2)$, i.e.,

$$(4.8) \quad \frac{n}{2} > 1 - t_1^{-1} = 1 + b(n,k),$$

then by (4.6) for any real number a satisfying

$$(4.9) \quad -\frac{2}{n-2} > -\frac{1}{a-1} > t_1, \quad \text{namely,} \quad \frac{n}{2} > a > 1 + b(n, k),$$

we have that $\sigma_k(A) \geq 0$ and that $\sigma_1(A)$ changes signs; more precisely, for a fixed $a \in (1 + b(n, k), n/2)$,

$$\begin{aligned} \sigma_1(A) &> 0, & \text{for } \frac{n-2}{n}a < r < a+1, & \text{and} \\ \sigma_1(A) &< 0, & \text{for } a-1 < r < \frac{n-2}{n}a. \end{aligned}$$

Now notice that (4.8) is assured by Proposition 4.3 below. This finishes the proof. We remark that, when $k < n$, the same result holds if $a = 1 + b(n, k)$.

Proposition 4.3. *For each $n \geq k \geq 4$,*

$$\frac{n}{2} > 1 + b(n, k) = 1 + \frac{n-k}{k-1} + \frac{1}{k-1} \sqrt{\frac{(n-1)(n-k)}{k}}.$$

Proof. Observe that

$$\frac{n}{2} - 1 - b(n, k) = c(n, k) \left\{ \left[\frac{(k-3)^2}{4} - \frac{1}{k} \right] n + \left(k - 2 + \frac{1}{k} \right) \right\} > 0,$$

in which

$$c(n, k) = \frac{n}{k-1} \left(\frac{(k-3)n}{2} + 1 + \sqrt{\frac{(n-1)(n-k)}{k}} \right)^{-1} > 0,$$

for $n \geq k \geq 4$. □

Remark 4.4. By varying the parameter a in Example 4.1 and Example 4.2, we can also provide examples of closed hypersurfaces in \mathbb{R}^{n+1} , which satisfy that $\sigma_k(A) \geq 0$ but $\sigma_{k-1}(A)$ changes signs, for each $3 \leq k \leq n$. (This in particular answers a question raised by H. D. Cao, for k being even.) Therefore, Theorem 1 is optimal for the Huisken–Sinestrari conjecture.

5. POSITIVE MASS THEOREM FOR HYPERSURFACES

Throughout this section, we denote by M a complete non-compact, embedded, and orientable C^{n+1} -smooth hypersurface in \mathbb{R}^{n+1} , unless otherwise indicated. Let $M_0 \subset M$ be the set of geodesic points, i.e.,

$$M_0 = \{p \in M : A(p) = 0\},$$

where A is the induced second fundamental form on M . Let H and R be, respectively, the mean curvature and scalar curvature of M . We adopt the convention that $H = -\operatorname{div}_0 \nu$, where ν is a smooth unit normal vector field to M and div_0 is the Euclidean divergence operator. In the following, we often abbreviate $\{H > 0\} = \{q \in M : H(q) > 0\}$ and $\{H < 0\} = \{q \in M : H(q) < 0\}$. For a function $f(x) = f(x^1, \dots, x^n)$, we denote $f_i = \partial f / \partial x^i$, $f_{ij} = \partial^2 f / \partial x^i \partial x^j$ for all $i, j = 1, \dots, n$, and $Df = (f_1, \dots, f_n)$.

Definition 5.1. We say that $M \subset \mathbb{R}^{n+1}$ is a C^k hypersurface of finitely many regular ends if M satisfies the following conditions:

- (1) M is a complete non-compact, embedded, and orientable C^k hypersurface in \mathbb{R}^{n+1} ;
- (2) There is a compact subset $K \subset M$ so that $M \setminus K$ consists of finitely many components N_i , where each N_i is the graph of a function $f_{(i)}$ over the exterior of a bounded region in some hyperplane Π_i ;
- (3) If $\{x^1, \dots, x^n\}$ are coordinates in Π_i , we require $\lim_{|x| \rightarrow \infty} f_{(i)}(x) = a_i$, where a_i is either a bounded constant, $a_i = \infty$, or $a_i = -\infty$.

We refer N_i the ends of M . We say that an end N_i is asymptotic to the hyperplane Π_i if $a_i = 0$. By translation, whenever $|a_i| < \infty$, N_i is asymptotic to a hyperplane.

Example 5.2 ([2]). The Schwarzschild manifolds of mass $m > 0$ in dimension $n \geq 3$ have the expression

$$\left(\mathbb{R}^n \setminus \bar{B}, \left(1 + \frac{m}{2r^{n-2}} \right)^{4/(n-2)} \delta \right).$$

Here \bar{B} denotes a closed ball of radius $(m/2)^{1/(n-2)}$. An n -dimensional Schwarzschild manifold can be isometrically embedded into \mathbb{R}^{n+1} , as a spherically symmetric C^∞ hypersurface of two regular ends.

In Section 3, we have proved that a *closed* hypersurface with nonnegative scalar curvature is weakly mean convex (up to an orientation). Below, we generalize the result to complete non-compact hypersurfaces of finitely many regular ends.

Proof of Theorem 4. Suppose on the contrary that H changes signs. The condition $R \geq 0$ implies that the set of geodesic points, denoted by M_0 , is equal to $\{q \in M : H(q) = 0\}$. By Theorem 2, the connected component M'_0 of M_0 which separates $\{H > 0\}$ and $\{H < 0\}$ must be unbounded. Lemma 3.8 yields that M'_0 lies in a hyperplane, say $\{x^{n+1} = 0\}$, which is tangent to M at every point of M'_0 . Because M'_0 is unbounded, we can conclude that an end N of M is asymptotic to $\{x^{n+1} = 0\}$. That is, N is the graph of a function f over the exterior of a bounded region in $\{x^{n+1} = 0\}$ and $|f(x)| \rightarrow 0$ as $|x| \rightarrow \infty$. Notice that f is not identically zero. Denote by $h = x^{n+1}|_M$ the height function on M . By Morse–Sard theorem, the level set $h^{-1}(\epsilon)$ is a C^{n+1} submanifold for almost every ϵ .

Let ν be the unit normal vector field on M which is pointing upward on N , i.e.,

$$(5.1) \quad \nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}} \quad \text{at } (x, f(x)) \in N.$$

Let H be the mean curvature with respect to ν . By Proposition 3.1, for $\epsilon > 0$ sufficiently small, $h^{-1}(\epsilon) \cap \{H > 0\} \neq \emptyset$. More importantly, we have $H \geq 0$ on each component of $h^{-1}(\epsilon)$ which has non-empty intersection with

$\{H > 0\}$, because H can only change signs through an unbounded subset of M contained in a hyperplane that is tangent to M by Theorem 2.

Notice that $h^{-1}(\epsilon)$ is closed for $0 < \epsilon \ll 1$. Let Σ_ϵ be the *outermost* connected component of $h^{-1}(\epsilon)$ such that $\Sigma_\epsilon \cap \{H > 0\} \neq \emptyset$, i.e., it is not enclosed by any other component of $h^{-1}(\epsilon)$ which has nonempty intersection with $\{H > 0\}$.

Now we fix a sufficiently small $\epsilon > 0$ so that Σ_ϵ has nonempty intersection with N and $|\nabla^M h| \neq 0$ on every point in Σ_ϵ . We caution readers that Σ_ϵ may not completely lie in N . Denote by η the unit normal vector on Σ_ϵ , pointing inward to the region enclosed by Σ_ϵ . Then, $\eta = Df/|Df|$ on $\Sigma_\epsilon \cap N$ because f tends to zero at infinity and Σ_ϵ is outermost. Thus, $\langle \nu, \eta \rangle < 0$ on $\Sigma_\epsilon \cap N$. Because $|\nabla^M h| \neq 0$ on every point in Σ_ϵ , $\langle \nu, \eta \rangle$ remains strictly negative on the rest part of Σ_ϵ . Denote by H_{Σ_ϵ} the mean curvature with respect to η . Apply Corollary 2.3 to obtain $H_{\Sigma_\epsilon} \leq 0$. This contradicts the compactness of Σ_ϵ (for, a compact set has at least one convex point at which $H_{\Sigma_\epsilon} > 0$). Therefore, H has a sign on M . \square

Corollary 5.3. *Let M be a connected C^{n+1} hypersurface in \mathbb{R}^{n+1} of finitely many regular ends with nonnegative scalar curvature. Suppose that an end N of M is asymptotic to the hyperplane Π , and N is the graph of a function over the exterior of an open ball in Π . Then N strictly lies in one side of Π , unless M is identical to Π .*

Proof. Suppose that N is the graph of a function f over $\{x^{n+1} = 0\} \setminus B_{r_1}$ for some $r_1 > 0$ and $|f(x)| \rightarrow 0$ as $|x| \rightarrow \infty$. We would like to prove that N is contained in either $\{x^{n+1} > 0\}$ or $\{x^{n+1} < 0\}$, unless N is identical to $\{x^{n+1} = 0\}$.

By Theorem 4, the mean curvature of M has a sign. Suppose that $H \geq 0$ with respect to ν , where ν is the upward pointing unit normal vector on N given by (5.1). (Otherwise, we reflect M about $\{x^{n+1} = 0\}$.) By Lemma 3.6,

$$\max_{\overline{B_{r_2}} \setminus B_{r_1}} f = \max_{\partial B_{r_2}} f \quad \text{for all } r_2 > r_1.$$

Because $|f(x)| \rightarrow 0$ as $|x| \rightarrow \infty$, we then conclude that $f \leq 0$ outside B_{r_1} . Moreover, by applying the strong maximum principle to $H(f) \geq 0$, we have $f < 0$ outside B_{r_1} , unless $f \equiv 0$. In the latter case, we can further conclude that M is identical to $\{x^{n+1} = 0\}$ by repeating the argument over $B_{r_2} \setminus B_{r_0}$ for $0 \leq r_0 < r_1$. \square

Next, we need a nice observation, due to Lam [13], that the scalar curvature has a divergence form for graphical hypersurfaces. This was originally proved by using the intrinsic definition of the scalar curvature [13]. We provide below a different proof which makes use of the extrinsic geometry.

Proposition 5.4. *If Ω is an open subset in \mathbb{R}^n . Let $f \in C^2(\Omega)$. Then the scalar curvature of the graph of f is*

$$(5.2) \quad R(f) = \sum_j \partial_j \sum_i \left(\frac{f_{ii}f_j - f_{ij}f_i}{1 + |Df|^2} \right).$$

Proof. By Gauss equation,

$$R = 2\sigma_2(A) = \sum_{i,j} (A_i^i A_j^j - A_i^j A_j^i),$$

in which $A = (A_j^i)$ is the shape operator with

$$A_j^i = \partial_j \left(\frac{f_i}{w} \right) = \left(\frac{f_i}{w} \right)_j, \quad \text{where } w = \sqrt{1 + |Df|^2}.$$

Observe that

$$\begin{aligned} A_i^i A_j^j &= \partial_j \left[\left(\frac{f_i}{w} \right)_i \frac{f_j}{w} \right] - \frac{f_j}{w} \left(\frac{f_i}{w} \right)_{ij}, \\ A_i^j A_j^i &= \partial_i \left[\left(\frac{f_i}{w} \right)_j \frac{f_j}{w} \right] - \frac{f_j}{w} \left(\frac{f_i}{w} \right)_{ij}. \end{aligned}$$

It follows that

$$\begin{aligned} R(f) &= \sum_j \partial_j \sum_i \left[\left(\frac{f_i}{w} \right)_i \frac{f_j}{w} - \left(\frac{f_j}{w} \right)_i \frac{f_i}{w} \right] \\ &= \sum_j \partial_j \sum_i \left(\frac{f_{ii}f_j - f_{ij}f_i}{w^2} \right). \end{aligned}$$

□

Definition 5.5 (cf. [13]). *Let M be a C^2 hypersurface of finitely many regular ends. Let N be one end of M , which is the graph of f over the exterior of a bounded region in the hyperplane Π . The mass of N is defined by*

$$(5.3) \quad m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{S_r} \frac{1}{1 + |Df|^2} \sum_{i,j} (f_{ii}f_j - f_{ij}f_i) \frac{x^j}{|x|} d\sigma,$$

where $S_r = \{(x^1, \dots, x^n) \in \Pi : |x| = r\}$, $d\sigma$ is the standard spherical volume measure of S_r , and $\omega_{n-1} = \text{vol}(\mathbb{S}^{n-1})$.

Lemma 5.6. *Let M be a C^2 hypersurface of finitely many regular ends. Let N be an end of M , which is the graph of f over the exterior of a bounded region in the hyperplane Π . If there exists a bounded region Ω_r in Π such that $\partial\Omega_r$ is the disjoint union of S_r and $\Sigma = \{x \in \Pi : f(x) = c\}$ for some constant c , and that $|Df|$ does not vanish on Σ , then*

$$m = \frac{1}{2(n-1)\omega_{n-1}} \left(\int_{\Sigma} \frac{|Df|^2}{1 + |Df|^2} H_{\Sigma} d\sigma + \lim_{r \rightarrow \infty} \int_{\Omega_r} R(f) dx \right),$$

where $R(f)$ is the scalar curvature of the graph of f , η is the unit normal vector on Σ pointing away from Ω_r , and H_Σ is the mean curvature of Σ with respect to η .

Proof. Applying the divergence theorem to (5.2) over Ω_r yields

$$\begin{aligned} & \int_{S_r} \frac{1}{1 + |Df|^2} \sum_{i,j=1}^n (f_{ii}f_j - f_{ij}f_i) \frac{x^j}{|x|} d\sigma \\ &= \int_{\Omega_r} R(f) dx - \int_{\Sigma} \frac{1}{1 + |Df|^2} \sum_{i,j=1}^n (f_{ii}f_j - f_{ij}f_i) \eta^j d\sigma. \end{aligned}$$

Because Σ is a level set of f , η equals either $Df/|Df|$ or $-Df/|Df|$. If $\eta = -Df/|Df|$,

$$\begin{aligned} H_\Sigma &= -\operatorname{div}_0 \eta = \sum_{i=1}^n \frac{\partial}{\partial x^i} \left(\frac{f_i}{|Df|} \right) \\ &= \frac{1}{|Df|^3} \sum_{i,j=1}^n (f_{ii}f_jf_j - f_{ij}f_if_j). \end{aligned}$$

We then derive

$$\begin{aligned} & \int_{S_r} \frac{1}{1 + |Df|^2} \sum_{i,j=1}^n (f_{ii}f_j - f_{ij}f_i) \frac{x^j}{|x|} d\sigma \\ (5.4) \quad &= \int_{\Omega_r} R(f) dx + \int_{\Sigma} \frac{|Df|^2}{1 + |Df|^2} H_\Sigma d\sigma. \end{aligned}$$

If $\eta = Df/|Df|$, we also derive the same identity. Letting $r \rightarrow \infty$, we prove the lemma. \square

Generally, the set Ω_r may not exist. We shall prove that if M has non-negative scalar curvature, then such Ω_r exists, and moreover $H_\Sigma \geq 0$.

Proof of Theorem 5. We assume that M is not a hyperplane; otherwise the theorem trivially holds. Consider an end N of M , and suppose that N is the graph of f over $\{x^{n+1} = 0\} \setminus B_{r_1}$ for some $r_1 > 0$. By Theorem 4, H has a sign on M . We may without loss of generality assume that $H \geq 0$ with respect to ν , where ν is the upward unit normal to N , given by

$$\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.$$

(Otherwise, we may replace f by $-f$.) We divide the proof into the following cases.

Case 1: $\lim_{|x| \rightarrow \infty} f(x) = a$ for some bounded constant a . By translation, we may assume that $|f(x)| \rightarrow 0$ as $|x| \rightarrow \infty$; namely, N is asymptotic to the hyperplane $\{x^{n+1} = 0\}$. By proof of Corollary 5.3, $N \subset \{x^{n+1} < 0\}$. Denote by $h = x^{n+1}|_M$ the height function on M . For $\epsilon > 0$ small enough, a

connected component of the level set $h^{-1}(-\epsilon)$ is contained in N . We define $\Sigma_{-\epsilon}$ to be an outermost connected component of the subset $\{x \in \{x^{n+1} = 0\} : f(x) = -\epsilon\}$. By Morse–Sard theorem, $\Sigma_{-\epsilon}$ is C^{n+1} for almost every ϵ . Moreover, because f tends to zero, for some small $\epsilon > 0$, $\eta = -Df/|Df|$ is the unit vector on $\Sigma_{-\epsilon}$, pointing inward to the bounded region in $\{x^{n+1} = 0\}$ enclosed by $\Sigma_{-\epsilon}$. Let $H_{\Sigma_{-\epsilon}}$ be the mean curvature of $\Sigma_{-\epsilon}$ defined by η . Then, $H_{\Sigma_{-\epsilon}} \geq 0$ by Corollary 2.3 and by $H \geq 0$. Applying Lemma 5.6, we have $m \geq 0$.

If $m = 0$, then M must be identical to $\{x^{n+1} = 0\}$. For, otherwise, there exists some positive ϵ so that $\Sigma_{-\epsilon}$ is contained in $\{x^{n+1} = 0\}$ with $H_{\Sigma_{-\epsilon}} \equiv 0$ by (5.4). This contradicts compactness of $\Sigma_{-\epsilon}$.

Case 2: $\lim_{|x| \rightarrow \infty} f(x) = \infty$. The set $\{x \in \{x^{n+1} = 0\} : f(x) = \Lambda\}$ lies in $\{x^{n+1}\} \setminus B_{r_1}$ for $\Lambda \gg 1$. Let Σ_{Λ} be the outmost component of the above set. For Λ sufficiently large, $\eta = -Df/|Df|$ is the normal vector to Σ_{Λ} pointing inward to the bounded region enclosed by Σ_{Λ} . Let H_{Λ} be the mean curvature with respect to η . Hence, $H_{\Sigma_{\Lambda}} \geq 0$ by Corollary 2.3 and by $H \geq 0$. Applying Lemma 5.6, we have $m \geq 0$. If $m = 0$, we can show that M is identical to a hyperplane as in Case 1.

Case 3: $\lim_{|x| \rightarrow \infty} f(x) = -\infty$. This case cannot happen. Otherwise, for some $\Lambda \gg 1$, there is a closed submanifold $\Sigma_{-\Lambda} \subset \{x \in \{x^{n+1} = 0\} \setminus B_{r_1} : f(x) = -\Lambda\}$ so that the unit normal vector $\eta = Df/|Df|$ is pointing inward to the region enclosed by $\Sigma_{-\Lambda}$. Let $H_{\Sigma_{-\Lambda}}$ be the mean curvature with respect to η . Then, $H_{\Sigma_{-\Lambda}} \leq 0$ by Corollary 2.3. This contradicts compactness of $\Sigma_{-\Lambda}$. □

Last, we verify below that our definition of the mass (5.3) coincides with the classical definition of the ADM mass, if we assume certain decay condition on the derivatives of f . Let us recall the definition of the ADM mass (see, for example, [1, Equation (4.1)]).

Definition 5.7. *We say that an n -dimensional manifold (M, g) has an asymptotically flat end N if $N \subset M$ is diffeomorphic to $\mathbb{R}^n \setminus B_1$ and N has a coordinate chart $\{y\}$ so that $g_{ij}(y) = \delta_{ij} + O_2(|y|^{-q})$ for some $q > (n-2)/2$. The O_2 indicates that first and second derivatives also decay at rates one and two orders faster, respectively.*

For $n \geq 3$, the ADM mass of the asymptotically flat end N is defined by

$$(5.5) \quad \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{|y|=r} \sum_{i,j} \left(\frac{\partial g_{ij}}{\partial y^i} - \frac{\partial g_{ii}}{\partial y^j} \right) \tau^j d\sigma_g,$$

where τ is the outward unit normal to $\{|y| = r\}$ with respect to g , $d\sigma_g$ is the volume measure of $\{|y| = r\}$ with respect to g , and $\omega_{n-1} = \text{vol}(\mathbb{S}^{n-1})$.

Lemma 5.8 (cf. [13]). *Let $n \geq 2$ and M a complete n -dimensional C^3 hypersurface of finitely many regular ends. Let N be an end of M which is*

the graph of f . Suppose that $R \in L^1(N)$ and $|Df(x)| = O(1)$ as $|x| \rightarrow \infty$. Then, the mass of N defined by (5.3) is finite.

If in addition $n \geq 3$, $|Df(x)|^2 = O_2(|x|^{-q})$ for some $q > (n-2)/2$, and $|Df(x)|^2|D^2f(x)| = o(|x|^{1-n})$ as $|x| \rightarrow \infty$, then (5.3) equals (5.5).

Proof. Applying the divergence theorem to (5.2) yields

$$2(n-1)\omega_{n-1}m = \int_{S_{r_0}} \frac{1}{1+|Df|^2} \sum_{i,j=1}^n (f_{ii}f_j - f_{ij}f_i) \frac{x^j}{|x|} d\sigma + \lim_{r \rightarrow \infty} \int_{r_0 \leq |x| \leq r} R(f) dx.$$

Because R is integrable over N and $|Df(x)| = O(1)$ as $|x| \rightarrow \infty$, the second term on the right hand side is bounded. Therefore, m is bounded.

To prove the second statement, we consider the coordinate chart $\{y\}$ of N , where

$$y^i = (0, \dots, \underbrace{x^i}_{i\text{-th}}, \dots, 0, f(0, \dots, \underbrace{x^i}_{i\text{-th}}, \dots, 0)).$$

Then

$$\frac{\partial}{\partial y^i} = \partial_i + f_i \partial_{n+1},$$

where we denote $\partial_i = \frac{\partial}{\partial x^i}$. Moreover, at the point $(x, f(x)) \in N$,

$$g_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle = \delta_{ij} + f_i f_j,$$

$$\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial (f_i f_j)}{\partial x^k} = f_{ik} f_j + f_i f_{jk}.$$

Therefore, N is an asymptotically flat end of M by hypothesis of $|Df|$. Denote by $\mu = \sum_{i=1}^n \frac{x^i}{r} \partial_i$ the outer unit normal to S_r in $\{x^{n+1} = 0\}$. Let τ be the outer unit normal to the graph of f over S_r in M . Then,

$$\tau = \frac{\mu + \mu(f) \partial_{n+1}}{\sqrt{1 + |\mu(f)|^2}},$$

and hence,

$$\tau^j = g(\tau, \frac{\partial}{\partial y^j}) = \left\langle \tau, \frac{\partial}{\partial y^j} \right\rangle = \frac{\mu^j + \mu(f) f_j}{\sqrt{1 + |\mu(f)|^2}}.$$

It follows that

$$(5.6) \quad \begin{aligned} & \int_{f(S_r)} \sum_{i,j} \left(\frac{\partial g_{ij}}{\partial y^i} - \frac{\partial g_{ii}}{\partial y^j} \right) \tau^j d\sigma_g \\ &= \int_{S_r} \sum_{i,j} (f_{ii}f_j - f_{ij}f_i) \frac{\mu^j + \mu(f) f_j}{\sqrt{1 + |\mu(f)|^2}} \sqrt{1 + |D^T f|^2} d\sigma, \end{aligned}$$

where $d\sigma_g$ and $d\sigma$ are the $(n-1)$ -Hausdorff measures on $f(S_r)$ and S_r respectively, and $D^T f$ is the derivative along the directions tangent to S_r . By [1, Proposition 4.1] and hypotheses on the derivatives of f , the left hand

side of (5.6) converges to (5.5) as $r \rightarrow \infty$, and the right hand side of (5.6) converges to (5.3). \square

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